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COMMENT

Self-avoiding walks on Sierpinski carpets

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Abstract. We calculate the fractal dimensions of self-avoiding walks (SAW) on Sierpinski carpets with the bond-moving-type renormalisation. The results suggest that carpets with the same fractal dimension of the random walk have almost the same fractal dimension of SAW within the approximation.

The self-avoiding walk (SAW) model represents a random walk that must not contain self-intersections. SAW has an averaged square distance $\langle r^2(t)\rangle$ as a function of time t:

$$\langle r^2(t)\rangle \sim t^{2/d_{\mathrm{SAW}}} \tag{1}$$

where d_{SAW} is the fractal dimension of SAW. Flory (1953) has predicted

$$d_{\mathrm{SAW}} = \tfrac{1}{3}(2+d) \tag{2}$$

where d is the dimension of the lattice. His prediction has agreed with several numerical calculations (Caracciolo and Sokal 1987). Now, what is the functional form of d_{SAW} for fractal lattices? The best proposal is that of Dekeyser *et al* (1987):

$$d_{\mathrm{SAW}} = \frac{2+d_{\mathrm{f}}}{2(1+1/d_{\mathrm{w}})} \tag{3}$$

where d_{w} is the fractal dimension of the random walk (RW) and d_{f} is the fractal dimension of the fractal lattice. Equation (3) is applicable to the exact results: RW (Hilfer and Blumen 1984) and SAW (Elezović *et al* 1987)† on the Sierpinski gaskets. The Sierpinski gaskets, however, are finitely ramified fractals. Their physical properties (for example, critical phenomena) depend upon whether the ramification (Mandelbrot 1982) of a lattice is infinite or not. Is equation (3) suitable for infinitely ramified fractals? In particular, equation (3) requires that the carpets with the same fractal dimension of SAW have the same fractal dimension of RW. To confirm this equality, we calculate d_{SAW} of Sierpinski carpets—one of infinitely ramified fractals—with the bond-moving-type renormalisation group method.

We construct Sierpinski carpets (Gefen *et al* 1984) in the following way: consider a square of unit area and subdivide it into b^2 subsquares, out of which l^2 subsquares are cut. At first, we consider only the behaviour of the central cutout and b is restricted to an odd number. The iteration of the renormalisation group transformation generates two basic exchange variables: the fugacities P and P_{w}. P is associated with a step

† It was pointed out by the referee that Dhar (1978) treated this problem first.

along the bond between two non-eliminated subsquares. P_w corresponds to a step along the bond which borders an eliminated subsquare. Following the study of da Silva and Droz (1987), we generate renormalised fugacities P' and P'_w. Figure 1 shows how to construct recursion relations. We define renormalised fugacities $\hat{P}$ on every bond. These renormalised fugacities $\hat{P}$ contain walks from a column to a neighbouring column through the bond. These walks consist of n ($n \leqslant (b-1)/2$) sequential vertical steps and a horizontal step. Owing to the anisotropy of the carpets, these walks, however, are of two different types. Hence we define $\hat{P}$ as a product of two types and take the geometric mean after bond-moving renormalisation. (One example is shown in figure 1 and its explanation is in figure 2. For the example shown, $\hat{P} = (P+2P^2+2P^3+2P^4) \times (P+2PP_w+PP_w^2+P^2P_w+P^2P_w^2+P^3P_w)$.) We can remark that, for the renormalisation invariance, when the bond neighbours the boundary of a $b \times b$ cell, the vertical steps on the boundary bonds are ignored (the boundary bonds in figure 1, therefore, are not illustrated). Hence, for the bond marked by # in figure 1, $\hat{P} = P(P+2P^2+2P^3+2P^4)$. Next, using bond-moving renormalisation, we construct $\tilde{P}_i$ ($i = 1, \ldots, b$); $\tilde{P}_i$ represents column to column fugacity. $\hat{P}$ values on the bond marked by an open circle are renormalised in $\tilde{P}_i$, and $\hat{P}$ on the bond marked by a full circle are renormalised in $\tilde{P}_{wi}$. $\tilde{P}_i$ and $\tilde{P}_{wi}$ are defined as a root of one bth of summation of

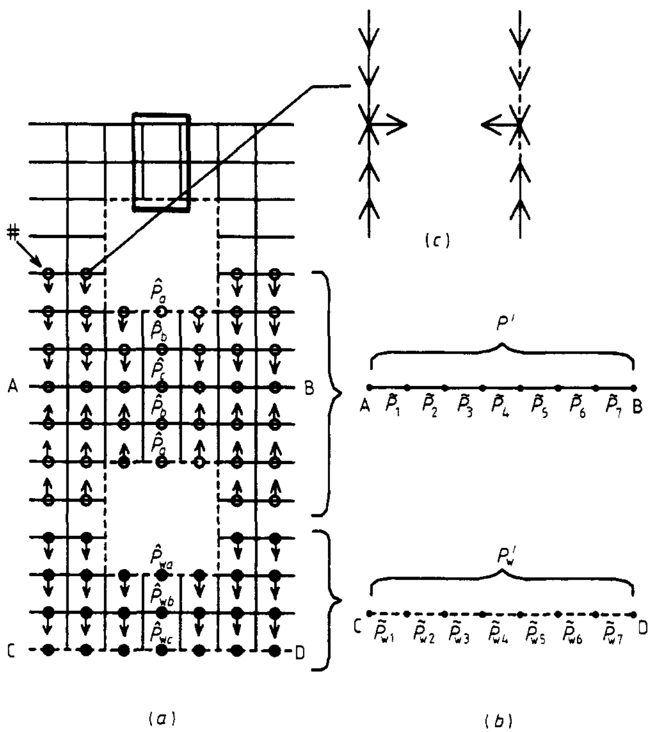

Figure 1. The carpet-type cells with $b = 7$, $l = 3$. Figure 1 shows the b times enlarged part surrounded by a bold rectangle. Full lines represent P bonds and broken lines represent P_w bonds. $\hat{P}$ on the bonds marked by an open circle are renormalised in P'. P'_w includes those through the bonds marked by a full circle. (a) Directions of bond moving. The arrows show the directions. (b) $\tilde{P}_i$ between A and B (above) and $\tilde{P}_{wi}$ between C and D (below), which are produced after bond moving. (c) The example of two types of walks through a bond. The explanation of these symbols is given in figure 2.

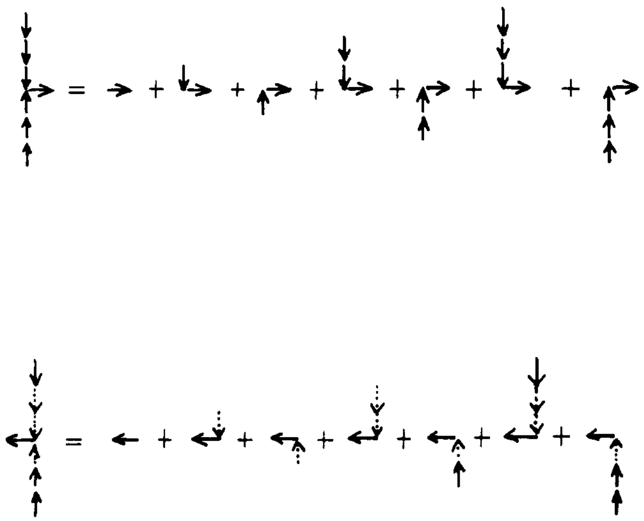

Figure 2. The explanation of the symbols in figure 1(c). This means n ($n \leqslant (b-1)/2$) sequential vertical steps and a horizontal step. Multiplying one of the two symbols by another, we obtain the renormalised fugacity $\hat{P}$.

$\hat{P}$ over moved bonds. For example, in figure 1,

$$\tilde{P}_4 = [\tfrac{1}{7}(2\hat{P}_a + 2\hat{P}_b + \hat{P}_c)]^{1/2} \tag{4a}$$

$$\tilde{P}_{w4} = [\tfrac{1}{7}(\hat{P}_{wa} + \hat{P}_{wb} + \hat{P}_{wc})]^{1/2} \tag{4b}$$

etc. The power of $\frac{1}{2}$ represents the geometric mean mentioned above. Finally, P' (P'_w) are obtained by multiplying $\tilde{P}_i$ ($\tilde{P}_{wi}$):

$$P' = \prod_{i=1}^{b} \tilde{P}_i \tag{5a}$$

and

$$P'_w = \prod_{i=1}^{b} \tilde{P}_{wi}. \tag{5b}$$

Thus we have obtained the recursion relations: $(P, P_w) \to (P', P'_w)$.

With these recursion relations, we draw flow diagrams (see figure 3). For $b = l+2$, the flow diagrams on the (P, P_w) plane have seven fixed points. The first four fixed points are trivial ones, A: $(0, 0)$, B: $(0, \infty)$, C: (∞, ∞), D: $(\infty, 0)$. To find the non-trivial fixed points, we next discuss flows along special axes. Starting at a point $(0, P_w)$ on the P_w axis ($P = 0$), the flow stays on this axis. On this axis, the flows near A go toward A and those near B go toward B. Hence, there is an additional fixed point on the $P = 0$ line, unstable in the direction of this axis. We denote this point by E. In the same manner, we examine the flow on the $P_w = 0$ line and find one more unstable fixed point, called G. For both fixed points, E and G, there exists a flow going out of the inside of the (P, P_w) plane into the fixed points. Consequently, there must be a final fixed point inside the plane. This fixed point, F, is unstable in all directions and provides the value of d_{SAW} later. For $b > l+2$, the flow diagrams do not include the fixed point D and do not have the fixed point G on the $P_w = 0$ line but inside the

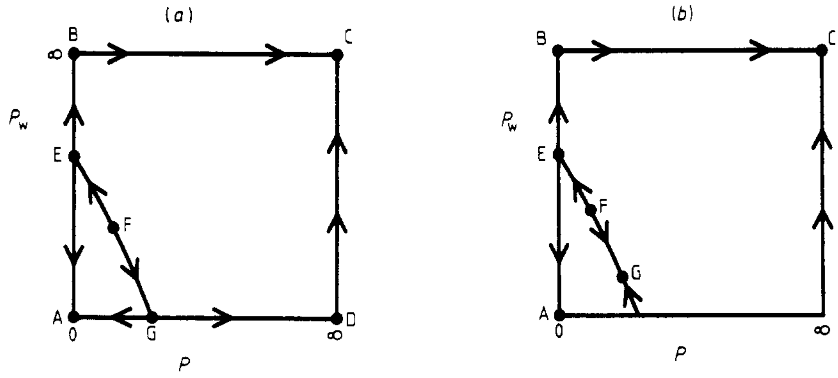

Figure 3. The flow diagrams on the (P, P_w) plane. (a) $b=l+2$, (b) $b>l+2$. They have seven (for (a)) or six (for (b)) fixed points. The fixed point F gives d_{SAW}.

(P, P_w) plane. Table 1 gives the numerical results for various b and l values. Last we calculate values of d_{SAW} from the recursion relation of the fugacities, according to Given and Mandelbrot (1983); they have calculated d_w from the recursion relation of the hopping probability. For this purpose, we define times T and T_w which are step times on P and P_w bonds, respectively. If we expand the renormalised fugacity P' and P'_w, the coefficient of the term $P^m P_w^n$ approximately yields the number of walks of this type consisting of m steps along the P bond and n steps along the P_w bond. Thus defining $Q = TP\partial/\partial P + T_w P_w \partial/\partial P_w$, we have $T' = (1/P')QP'$ and $T'_w = (1/P'_w)QP_w$, where the primed 'times' are renormalised step times. These provide the recursion relation for T/T_w:

$$\frac{T'}{T'_w} = \left(\frac{T}{T_w}\frac{\partial P'}{\partial P} + \frac{P_w}{P'}\frac{\partial P'}{\partial P_w}\right)\left(\frac{T}{T_w}\frac{P}{P'_w}\frac{\partial P'_w}{\partial P} + \frac{\partial P'_w}{\partial P_w}\right)^{-1} \tag{6}$$

where the fugacities and the differentiations take values at the fixed point F. The fixed point of (6) is given as

$$\frac{T}{T_w} = \left\{\frac{\partial P'}{\partial P} - \frac{\partial P'_w}{\partial P_w} + \left[\left(\frac{\partial P'}{\partial P} - \frac{\partial P'_w}{\partial P_w}\right)^2 + 4\frac{\partial P'}{\partial P_w}\frac{\partial P'_w}{\partial P}\right]^{1/2}\right\}\left(2\frac{P}{P'_w}\frac{\partial P'_w}{\partial P}\right)^{-1}. \tag{7}$$

From equation (1), d_{SAW} takes the form

$$d_{SAW} = \frac{\ln(T'/T)}{\ln b} \tag{8a}$$

Table 1. The numerical results for the carpets with central cutout. The values of d_{SAW} monotonically decrease with decreasing d_f.

b	l	d_f	P^*	P_w^*	d_{SAW}
5	1	1.975	0.335 23	1.3611	1.256
9	3	1.946	0.279 085	1.6206	1.241
7	3	1.896	0.212 21	1.5381	1.228
3	1	1.892	0.195 3	1.236	1.207
5	3	1.723	0.043 63	1.5245	1.151
7	5	1.633	0.013 056	1.7027	1.140
9	7	1.577	0.004 882	1.8349	1.137

or

$$d_{\mathrm{SAW}}=\frac{\ln(T'_{\mathrm{w}}/T_{\mathrm{w}})}{\ln b} \tag{8b}$$

where

$$\frac{T'}{T}=\frac{\partial P'}{\partial P}+\frac{T_{\mathrm{w}}}{T}\frac{P_{\mathrm{w}}}{P'}\frac{\partial P'}{\partial P_{\mathrm{w}}} \tag{9a}$$

and

$$\frac{T'_{\mathrm{w}}}{T_{\mathrm{w}}}=\frac{T}{T_{\mathrm{w}}}\frac{P}{P'_{\mathrm{w}}}\frac{\partial P'_{\mathrm{w}}}{\partial P}+\frac{\partial P'_{\mathrm{w}}}{\partial P_{\mathrm{w}}}. \tag{9b}$$

In (9*a*) and (9*b*) T/T_{w} is given in (7). Table 1 shows the numerical results of d_{SAW}, which monotonically decrease with decreasing d_{f}. This tendency qualitatively agrees with the following equation:

$$d_{\mathrm{SAW}}=\tfrac{1}{3}(2+d_{\mathrm{f}}). \tag{10}$$

We can obtain (10) by setting $d_{\mathrm{w}}=2$ in (3) because of the small deviation of d_{w} from 2 on the Sierpinski carpet (refer to the equation in the caption of table 2). Hence our approximation can show the dependence of d_{SAW} upon d_{f}. Our results, however, are not accurate enough to check (3) directly. In fact, substituting our results into (3), we get values of d_{w} *less than* 2: impossible values!

Next we consider SAW on various carpets. The bond-moving renormalisation tells us that the value of d_{w} does not depend upon the shapes: the behaviour of the cutout of l^2 eliminated subsquares (Taguchi 1988). Equation (3) also tells us that, if two carpets have the same value of d_{w}, d_{SAW} of these carpets are equal. It is interesting to confirm this equality directly. Fortunately, the carpets can take different shapes without varying the value of b and l, i.e. d_{w}. Figure 4 shows some examples of carpets which have the same values of d_{w} but have different shapes. Table 2 shows the numerical results for these carpets. The carpets of different shapes seem to have different values of d_{SAW}, even if they have the same value of d_{w}. To estimate the degree of error, we calculate d_{SAW} of the two-dimensional ordinary lattice. To this end, we set $l=0$ in our formula. Then we get the recursion relation

$$P'=P(P+2P^2+\ldots+2P^{(b+1)/2})^{b-1}. \tag{11}$$

Because $Q=TP\partial/\partial P$, $T'/T=\partial P'/\partial P$. Substituting the fixed-point values into (10*a*), we obtain the values of d_{SAW} (see table 3). Table 3 shows that the degree of error is about a few per cent. On the other hand, d_{SAW} of the carpet with the same d_{w} fluctuates over less than a few per cent. Hence we cannot distinguish between the error and the effect of different shapes. However, the carpets with the same d_{w} turn out to have almost the same d_{SAW} within this approximation.

To conclude, we propose a new renormalisation method for the self-avoiding walk. This provides the value of the fractal dimension of self-avoiding walks, d_{SAW}. The degree of error of d_{SAW} is less than a few per cent for SAW on the ordinary $d=2$ lattice. Applying this method to SAW on Sierpinski carpets, we obtain d_{SAW} of the carpets of several fractal dimensions, d_{f}. We also examine which carpets have the same d_{SAW}. It turns out that the carpets of same d_{w} have almost the same d_{SAW}.

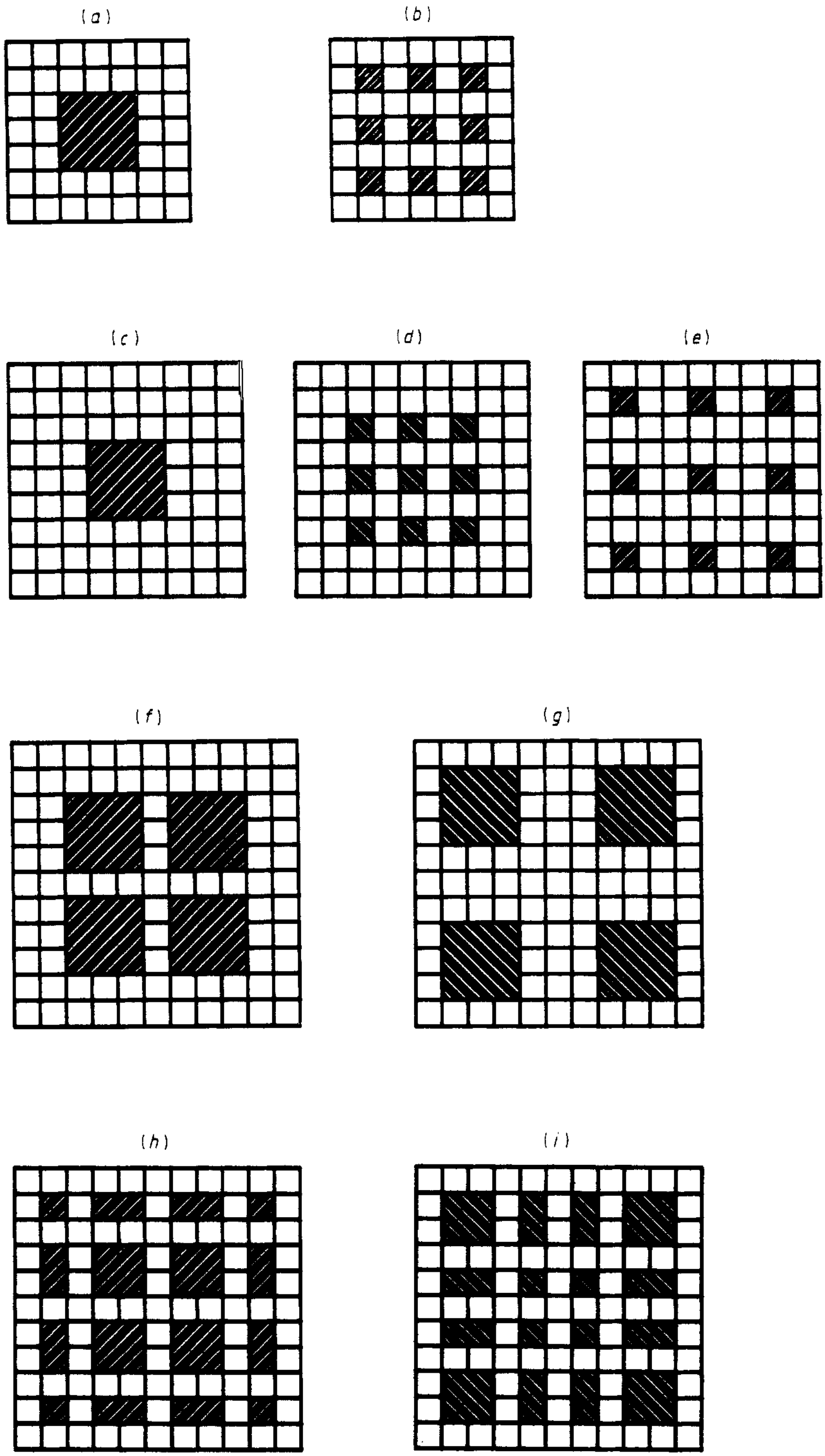

Figure 4. Some examples of the set of carpets with the same value of d_w (see table 2). The groups (a) and (b), (c)–(e) and (f)–(i) have the same value of d_w.

Table 2. Numerical results for the carpets with the same d_w and different shapes (see figure 3). d_w is calculated with the bond-moving renormalisation (Taguchi 1988), $d_w = \ln(\{[(b-l)/b]+[l/(b-l)]\}(b^2-l^2))/\ln b$.

		F			
Figure 4	d_f	P^*	P_w^*	d_{SAW}	d_w
(*a*)	1.896	0.212 21	1.5381	1.228	2.039
(*b*)		0.128 235	1.1931	1.231	
(*c*)	1.946	0.279 085	1.6206	1.241	2.017
(*d*)		0.197 905	1.3766	1.235	
(*e*)		0.218 94	1.3342	1.229	
(*f*)	1.853	0.102 466	1.5628	1.210	2.063
(*g*)		0.129 84	1.4554	1.230	
(*h*)		0.061 065	1.3120	1.199	
(*i*)		0.075 867	1.2586	1.215	

Table 3. Numerical results of $d = 2$ ordinary lattices. Setting $l = 0$ in our formula, we get the recursion relation for these lattices. b represents the length of a cell side. The results show that our method has error of a few per cent.

b	P^*	$\partial P'/\partial P$	d_{SAW}	error (%)
3	0.5	4	1.262	5.3
5	0.440 62	7.922	1.286	3.5
7	0.423 86	12.147	1.283	3.8
9	0.417 98	16.414	1.273	4.5
11	0.415 73	20.632	1.262	5.3
Reliable	0.3791†	—	$\frac{4}{3}$	—

† From numerical results (Caracciolo and Sokal 1987).

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